

Common Fixed Point Theorems for Three Selfmaps of a Complete S-Metric Space¹

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ABSTRACT

Suppose (X, S) is a S- metric space and P, Q and T are selfmaps of X . If these three maps and the space X satisfy certain conditions, we shall prove that they have a unique common fixed point in this paper. As a consequence we deduce a common fixed point theorem for three selfmaps of a complete S- metric space. Further, we show that a common fixed point theorem for three selfmaps of a metric space proved by S. L. Singh and S. P. Singh ([9]) follows as a particular case of the theorem.

Mathematics Subject Classification: 47H10, 54H25.

Key Words: S-metric space; Compatible; Fixed point theorem

1. INTRODUCTION AND PRELIMINARIES

It is well known that in applied mathematics, many of the most important nonlinear problems can be reduced to solving a given equation. This equation can be reformulated as finding the fixed point or zero of an operator, highlighting the importance of fixed point theory as a key part of nonlinear functional analysis. Fixed point theory has found wide applications in various disciplines, including nonlinear partial differential equations, nonlinear integral equations, control theory, optimization theory, economics, and engineering.

On the other hand, some authors are interested and have tried to give generalizations of metric spaces in different ways. In 1963 Gahler [6] gave the concepts of 2- metric space further in 1992 Dhage [2] modified the concept of 2- metric space and introduced the concepts of D-metric space also proved fixed point theorems for selfmaps of such spaces. Later researchers have made a significant contribution to fixed point of D- metric spaces in [1], [3], and [4]. Unfortunately almost all the fixed point theorems proved on D-metric spaces are not valid in view of papers [6], [7] and [8]. Sedghi et al. [10] modified the concepts of D- metric space and introduced the concepts of D*- metric space also proved a common fixed point theorems in D*- metric space.

Recently, Sedghi et al [11] introduced the concept of S- metric space which is different from other space and proved fixed point theorems in S-metric space. They also gives some examples of S- metric spaces which shows that S- metric space is different from other spaces. In fact they give following concepts of S- metric space.

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Definition 1.1([11]): Let X be a non-empty set. An S -metric space on X is a function

$S: X^3 \rightarrow [0, \infty)$ that satisfies the following conditions, for each $x, y, z, a \in X$

- (i) $S(x, y, z) \geq 0$
- (ii) $S(x, y, z) = 0$ if and only if $x = y = z$.
- (iii) $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$

The pair (X, S) is called an **S -metric space**.

Immediate examples of such S -metric spaces are:

Example 1.2: Let $\mathbb{R}$ be the real line. Then $S(x, y, z) = |x - y| + |y - z| + |z - x|$ for each

$x, y, z \in \mathbb{R}$ is an S -metric on $\mathbb{R}$. This S -metric is called the usual S -metric on $\mathbb{R}$.

Example 1.3: Let $X = \mathbb{R}^2$, d be the ordinary metric on X .

Put $S(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ is an S -metric on X . If we connect the points x, y, z by a line, we have a triangle and if we choose a point a mediating this triangle then the inequality $S(x, y, z) \leq S(x, x, a) + S(y, y, a) + S(z, z, a)$ holds. In fact

$$\begin{aligned} S(x, y, z) &= d(x, y) + d(y, z) + d(z, x) \\ &\leq d(x, a) + d(a, y) + d(y, a) + d(a, z) + d(z, a) + d(a, x) \\ &= S(x, x, a) + S(y, y, a) + S(z, z, a) \end{aligned}$$

Example 1. 4: Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X , then $S(x, y, z) = \|x - z\| + \|y - z\|$ is an S -metric on X .

Remark 1. 5: it is easy to see that every D^* -metric is S -metric, but in general the converse is not true, see the following example.

Example 1. 6: Let $X = \mathbb{R}^n$ and $\| \cdot \|$ a norm on X , then $S(x, y, z) = \|y + z - 2x\| + \|y - z\|$ is an S -metric on X , but it is not D^* -metric because it is not symmetric.

Lemma 1. 7: In an S -metric space, we have $S(x, x, y) = S(y, y, x)$.

Proof: By the third condition of S -metric, we get

$$S(x, x, y) \leq S(x, x, x) + S(x, x, x) + S(y, y, x) = S(y, y, x) \dots (1)$$

and similarly

$$S(y, y, x) \leq S(y, y, y) + S(y, y, y) + S(x, x, y) = S(x, x, y) \dots (2)$$

Hence, by (1) and (2), we obtain $S(x, x, y) = S(y, y, x)$.

Definition 1.8: Let (X, S) be an S-metric space. For $x \in X$ and $r > 0$, we define the open ball $B_S(x, r)$ and closed ball $B_S[x, r]$ with a center x and a radius r as follows

$$B_S(x, r) = \{y \in X; S(x, y, y) < r\}$$

$$B_S[x, r] = \{y \in X; S(x, y, y) \leq r\}$$

For example, Let $X = \mathbb{R}$. Denote $S(x, y, z) = |y + z - 2x| + |y - z|$ for all $x, y, z \in \mathbb{R}$. Therefore $B_S(1, 2) = \{y \in \mathbb{R}; S(y, y, 1) < 2\}$

$$= \{y \in \mathbb{R}; |y - 1| < 1\} = (0, 2).$$

Definition 1.9: Let (X, S) be an S-metric space and $A \subset X$.

(1) If for every $x \in A$, there is a $r > 0$ such that $B_S(x, r) \subset A$, then the subset A called an **open subset** of X

(2) If there is a $r > 0$ such that $S(x, x, y) < r$ for all $x, y \in A$ then A is said to be **S-bounded**.

(3) A sequence $\{x_n\}$ in X **converge to x** if and only if $S(x_n, x_n, x) \rightarrow 0$ as $n \rightarrow \infty$. That is for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $S(x_n, x_n, x) < \epsilon$ and we denote this by $\lim_{n \rightarrow \infty} x_n = x$

(4) A sequence $\{x_n\}$ in X is called a **Cauchy sequence** if for each $\epsilon > 0$, there exists $n_0 \in \mathbb{N}$ such that $S(x_n, x_n, x_m) < \epsilon$ for each $m, n \geq n_0$

(5) The S-metric space (X, S) is said to be **complete** if every Cauchy sequence is convergent sequence.

(6) Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exists $r > 0$ such that

$B_S(x, r) \subset A$. Then τ is a topology on X (induced by the S-metric S).

(7) If (X, τ) is a compact topological space we shall call (X, S) is a **compact** S-metric space.

Lemma 1. 10([11]): Let (X, S) be an S-metric space. If $r > 0$ and $x \in X$, then the open ball

$B_S(x, r)$ is an open subset of X .

Lemma 1. 11([11]): Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x ,

then x is unique.

Lemma 1. 12([11]): Let (X, S) be an S-metric space. If the sequence $\{x_n\}$ in X converges to x ,

then $\{x_n\}$ is a Cauchy sequence.

Lemma 1. 13([11]): Let (X, S) be an S-metric space. If there exists sequences $\{x_n\}$ and $\{y_n\}$ such

that $\lim_{n \rightarrow \infty} x_n = x$ and $\lim_{n \rightarrow \infty} y_n = y$, then $\lim_{n \rightarrow \infty} S(x_n, x_n, y_n) = S(x, x, y)$.

Lemma1. 14: Let (X, d) be a metric space. Then we have

1. $S_d(x, y, z) = d(x, y) + d(y, z) + d(z, x)$ for all $x, y, z \in X$ is an S-metric on X
2. $x_n \rightarrow x$ in (X, d) if and only if $x_n \rightarrow x$ in (X, S_d)
3. $\{x_n\}$ is a Cauchy sequence in (X, d) if and only if $\{x_n\}$ is a Cauchy sequence in (X, S_d)
4. (X, d) is complete if and only if (X, S_d) is complete

Proof: (1) See [Example (3), Page 260]

(2) $x_n \rightarrow x$ in (X, d) if and only if $d(x_n, x) \rightarrow 0$, if and only if $S_d(x_n, x_n, x) = 3d(x_n, x) \rightarrow 0$ that is, $x_n \rightarrow x$ in (X, S_d)

(3) $\{x_n\}$ is a Cauchy in (X, d) if and only if $d(x_n, x_m) \rightarrow 0$ as $n, m \rightarrow \infty$, if and only if $S_d(x_n, x_n, x_m) = 3d(x_n, x_m) \rightarrow 0$, $n, m \rightarrow \infty$, that is, $\{x_n\}$ is Cauchy in (X, S_d)

(4) It is a direct consequence of (2) and (3)

Notation: For any selfmap T of X , we denote $T(x)$ by Tx .

If P and Q are selfmaps of a set X , then any $z \in X$ such that $Pz = Qz = z$ is called a **common fixed point** of P and Q .

Two selfmaps P and Q of X are said to be **commutative** if $PQ = QP$ where PQ is their composition PoQ defined by $(PoQ)x = PQx$ for all $x \in X$.

Definition 1.15: Suppose P and Q are selfmaps of a S-metric space (X, S) satisfying the condition $Q(X) \subseteq P(X)$. Then for any $x_0 \in X$, $Qx_0 \in Q(X)$ and hence $Qx_0 \in P(X)$, so that there is a $x_1 \in X$ with $Qx_0 = Px_1$, since $Q(X) \subseteq P(X)$. Now $Qx_1 \in Q(X)$ and hence there is a $x_2 \in X$ with $Qx_1 \in Q(X) \subseteq P(X)$ so that $Qx_1 = Px_2$. Again $Qx_2 \in Q(X)$ and hence $Qx_2 \in P(X)$ with $Qx_2 = Px_3$. Thus repeating this process to each $x_0 \in X$, we get a sequence $\{x_n\}$ in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$. We shall call this sequence as an **associated sequence of x_0 relative to the two selfmaps P and Q** . It may be noted that there may be more than one associated sequence for a point $x_0 \in X$ relative to selfmaps P and Q .

Let P and Q are selfmaps of a S-metric space (X, S) such that $Q(X) \subseteq P(X)$. For any $x_0 \in X$, if $\{x_n\}$ is a sequence in X such that $Qx_n = Px_{n+1}$ for $n \geq 0$, then $\{x_n\}$ is called an **associated sequence** of x_0 relative to the two selfmaps P and Q .

Definition 1.16: A function $\emptyset: [0, \infty) \rightarrow [0, \infty)$ is said to be a **contractive modulus**, if $\emptyset(0) = 0$ and $\emptyset(t) < t$ for $t > 0$.

Definition 1.17: A real valued function $\emptyset$ defined on $X \subseteq \mathbb{R}$ is said to be **upper semi continuous**, if $\limsup_{n \rightarrow \infty} \emptyset(t_n) \leq \emptyset(t)$ for every sequence $\{t_n\}$ in X with $t_n \rightarrow t$ as $n \rightarrow \infty$.

Definition 1.18: If P and Q are selfmaps of a S-metric space (X, S) such that for every sequence $\{x_n\}$ in X with $\lim_{n \rightarrow \infty} Px_n = \lim_{n \rightarrow \infty} Qx_n = t$, we have

$\lim_{n \rightarrow \infty} S(PQx_n, QPx_n, QPx_n) = 0$, then we say that P and Q are **compatible**.

2. THE MAIN RESULTS:

2.1 Theorem: Let P, Q and T are selfmaps of a S- metric space (X, S) satisfying the conditions

- (i) $P(X) \cup Q(X) \subseteq T(X)$

$$(ii) \quad S(Px, Qy, Qy) \leq \emptyset (\lambda(x, y)) \text{ for all } x, y \in X$$

where $\emptyset$ is an upper semi continuous and contractive modulus

and

$$(ii)' \quad \lambda(x, y) = \max \{S(Tx, Ty, Ty), S(Px, Tx, Tx), S(Qy, Ty, Ty),$$

$$\frac{1}{2} [S(Px, Ty, Ty) + S(Qy, Tx, Tx)]\}$$

$$(iii) \text{ either } (P, T) \text{ or } (Q, T) \text{ are compatible pair}$$

and

$$(iv) \quad T \text{ is continuous}$$

Further, if

- (i) there is a point $x_0 \in X$ and an associated sequence $\{x_n\}$ of x_0 relative to the three selfmaps such that the sequence $Px_0, Qx_1, Px_2, Qx_3, \dots, Px_{2n}, Qx_{2n+1}, \dots$ converge to some point $z \in X$,

then P, Q and T have a unique common fixed point $z \in X$.

Before we give the proof of theorem, we establish some lemmas.

2.1. 1 Lemma: Suppose P, Q and T are selfmaps of a S - metric space (X, S) satisfying the conditions (i), (ii), (iv) and (v) of theorem 2.1. Then for any associated sequence $\{x_n\}$ of x_0 relative to P, Q and T we have

$$(a) \quad \lim_{n \rightarrow \infty} \lambda(Tx_{2n}, x_{2n+1}) = S(z, Tz, Tz) \text{ if } (P, T) \text{ is compatible}$$

and

$$(b) \quad \lim_{n \rightarrow \infty} \lambda(x_{2n}, Tx_{2n+1}) = S(z, Tz, Tz) \text{ if } (Q, T) \text{ is compatible}$$

Proof: Since by (v), each of the sequences $\{Px_{2n}\}$ and $\{Qx_{2n+1}\}$ converge to $z \in X$ and since $Px_{2n} = Tx_{2n+1}$ and $Qx_{2n+1} = Tx_{2n+2}$ for $n \geq 0$, we have

$$(2.1.2) \quad Px_{2n}, Qx_{2n+1}, Tx_{2n}, Tx_{2n+1} \rightarrow z \text{ as } n \rightarrow \infty.$$

Now since T is continuous, we have

$$(2.1.3) \quad TPx_{2n} \rightarrow Tz, T^2x_{2n} \rightarrow Tz \text{ as } n \rightarrow \infty$$

(a) If the pair the pair (P, T) is compatible, we have

$$(2.1.4) \quad \lim_{n \rightarrow \infty} S(PTx_{2n}, TPx_{2n}, TPx_{2n}) = 0$$

since $Px_{2n}, Tx_{2n} \rightarrow z$ as $n \rightarrow \infty$ by (2.1.2).

Now, in view of (2.1.3) and (2.1.4), we get

$$(2.1.5) \quad PTx_{2n} \rightarrow Tz \text{ as } n \rightarrow \infty.$$

Also, from (ii)', we have

$$(2.1.6) \quad \lambda(Tx_{2n}, x_{2n+1}) = \max \{S(T^2x_{2n}, Tx_{2n+1}, Tx_{2n+1}), S(PTx_{2n}, T^2x_{2n}, T^2x_{2n}),$$

$$S(Qx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2} [S(PTx_{2n}, Tx_{2n+1}, Tx_{2n+1}) + S(Qx_{2n+1}, T^2x_{2n}, T^2x_{2n})]\}$$

Letting $n \rightarrow \infty$ in (2. 1. 6) and using the continuity of S , (2. 1. 2), (2. 1. 3), (2. 1. 4) and

(2. 1. 5) we get

$$\begin{aligned}\lim_{n \rightarrow \infty} \lambda(Tx_{2n}, x_{2n+1}) &= \max \{S(Tz, z, z), S(Tz, Tz, Tz), S(z, z, z), \\ &\quad \frac{1}{2} [S(Tz, z, z) + S(z, Tz, Tz)] \} \\ &= S(z, Tz, Tz).\end{aligned}$$

(b) If the pair the pair (Q, T) is compatible, we have

$$(2. 1. 7) \lim_{n \rightarrow \infty} S(TQx_{2n+1}, QTx_{2n+1}, QTx_{2n+1}) = 0$$

in view of (2. 1. 2). Also since T is continuous, we have again by (2. 1. 2),

$$(2. 1. 8) T^2x_{2n+1} \rightarrow Tz \text{ and } TQx_{2n+1} \rightarrow Tz \text{ as } n \rightarrow \infty.$$

Now, in view of (2. 1. 7) and (2. 1. 8), we get

$$(2. 1. 9) QTx_{2n+1} \rightarrow Tz \text{ as } n \rightarrow \infty.$$

Now, from (ii)', we have

$$(2. 1. 10) \lambda(x_{2n}, Tx_{2n+1}) = \max \{S(Tx_{2n}, T^2x_{2n+1}, T^2x_{2n+1}), S(Px_{2n}, Tx_{2n}, Tx_{2n}),$$

$$S(QTx_{2n+1}, T^2x_{2n+1}, T^2x_{2n+1}), \frac{1}{2} [S(Px_{2n}, T^2x_{2n+1}, T^2x_{2n+1}) + S(QTx_{2n+1}, Tx_{2n}, Tx_{2n})]\}$$

Now letting $n \rightarrow \infty$ in (2. 1. 10) and using the continuity of S , (2. 1. 2), (2. 1. 8) and (2. 1. 9), we get

$$\begin{aligned}\lim_{n \rightarrow \infty} \lambda(x_{2n}, Tx_{2n+1}) &= \max \{S(z, Tz, Tz), S(z, z, z), S(Tz, Tz, Tz), \\ &\quad \frac{1}{2} [S(z, Tz, Tz) + S(z, Tz, Tz)] \} \\ &= S(z, Tz, Tz). \text{ Hence the lemma.}\end{aligned}$$

2. 2 Prof of Theorem 2. 1: In this section we first prove the existence of a common fixed point in the two cases of the condition (iii) in Theorem 2. 1.

Case(I). First suppose that the pair (P, T) is compatible. Then from (ii), we have

$$(2. 2. 1) S(PTx_{2n}, Qx_{2n+1}, Qx_{2n+1}) \leq \emptyset (\lambda(Tx_{2n}, x_{2n+1}))$$

In which on letting $n \rightarrow \infty$ using Lemma 2. 1. 1, and the continuity of S , we get

$$(2. 2. 2) S(Tz, z, z) \leq \emptyset (S(Tz, z, z))$$

and this leads to a contradiction if $Tz \neq z$. Therefore $Tz = z$.

Again, from condition (ii), we have

$$(2. 2. 3) S(Pz, Qx_{2n+1}, Qx_{2n+1}) \leq \emptyset (\lambda(z, x_{2n+1})). \text{ But}$$

$$\lambda(z, x_{2n+1}) = \max \{S(Tz, Tx_{2n+1}, Tx_{2n+1}), S(Pz, Tz, Tz),$$

$$S(Qx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2} [S(Pz, Tx_{2n+1}, Tx_{2n+1}) + S(Qx_{2n+1}, Tz, Tz)]\}$$

which on letting $n \rightarrow \infty$ and the use of continuity of S imply

$$\lim_{n \rightarrow \infty} \lambda(z, x_{2n+1}) = S(z, Pz, Pz),$$

Now letting $n \rightarrow \infty$ in (2. 2. 3), we get by the continuity of S that

$$(2. 2. 4) S(Pz, z, z) \leq \emptyset (S(Pz, z, z))$$

and this leads to a contradiction if $Pz \neq z$. Therefore $Pz = z$.

Now again, from condition (ii), we have

$$(2. 2. 5) S(Px_{2n}, Qz, Qz) \leq \emptyset (\lambda(x_{2n}, z)). \text{ But}$$

$$\lambda(x_{2n}, z) = \max \{S(Tx_{2n}, Tz, Tz), S(Px_{2n}, Tx_{2n}, Tx_{2n}), S(Qz, Tz, Tz),$$

$$\frac{1}{2} [S(Px_{2n}, Tz, Tz) + S(Qz, Tx_{2n}, Tx_{2n})]\}$$

in which on letting $n \rightarrow \infty$ and the continuity of S , we get

$$\lim_{n \rightarrow \infty} \lambda(x_{2n}, z) = S(z, Qz, Qz), \text{ since } Px_{2n} \rightarrow z, Tx_{2n} \rightarrow z \text{ as } n \rightarrow \infty. \text{ Then (2. 2. 5) gives}$$

$$(2. 2. 6) S(z, Qz, Qz) \leq \emptyset (S(z, Qz, Qz))$$

and this will give a contradiction if $Qz \neq z$. Therefore $Qz = z$.

Hence $z = Pz = Qz = Tz$, showing that z is a common fixed point of P, Q and T .

Case (ii): Suppose that the pair (Q, T) is compatible, then from (ii), we have

$$(2. 2. 7) S(Px_{2n}, QTx_{2n+1}, QTx_{2n+1}) \leq \emptyset (\lambda(x_{2n}, Tx_{2n+1}))$$

in which on letting $n \rightarrow \infty$ using Lemma 2. 1. 1, (2. 1. 9), and the continuity of S , we get

$$(2. 2. 8) S(z, Tz, Tz) \leq \emptyset (S(z, Tz, Tz))$$

and this will be a contradiction if $Tz \neq z$. Therefore $Tz = z$.

Again, from condition (ii), we have

$$(2. 2. 9) S(Pz, Qx_{2n+1}, Qx_{2n+1}) \leq \emptyset (\lambda(z, x_{2n+1})). \text{ But}$$

$$\lambda(z, x_{2n+1}) = \max \{S(Tz, Tx_{2n+1}, Tx_{2n+1}), S(Pz, Tz, Tz), S(Qx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}),$$

$$\frac{1}{2} [S(Pz, Tx_{2n+1}, Tx_{2n+1}) + S(Qx_{2n+1}, Tz, Tz)]\}$$

$$\text{so that } \lim_{n \rightarrow \infty} \lambda(z, x_{2n+1}) = S(z, Pz, Pz),$$

Therefore, from (2. 2. 9), we have

$$(2. 2. 10) S(Pz, z, z) \leq \emptyset (S(Pz, z, z))$$

and this leads to a contradiction if $Pz \neq z$. Therefore $Pz = z$.

Again, from condition (ii), we have

$$(2. 2. 11) S(Px_{2n}, Qz, Qz) \leq \emptyset (\lambda(x_{2n}, z)). \text{ But}$$

$$\lambda(x_{2n}, z) = \max \{S(Tx_{2n}, Tz, Tz), S(Px_{2n}, Tx_{2n}, Tx_{2n}), S(Qz, Tz, Tz),$$

$$\frac{1}{2} [S(Px_{2n}, Tz, Tz) + S(Qz, Tx_{2n}, Tx_{2n})]\}$$

in which on letting $n \rightarrow \infty$ we get $\lim_{n \rightarrow \infty} \lambda(x_{2n}, z) = S(z, Qz, Qz)$,

since $Px_{2n} \rightarrow z$, $Tx_{2n} \rightarrow z$ as $n \rightarrow \infty$ and $Pz = z = Tz$,

Then (2. 2. 11) gives

$$(2. 2. 12) S(z, Qz, Qz) \leq \Phi(S(z, Qz, Qz))$$

and this will give a contradiction if $Qz \neq z$. Therefore $Qz = z$.

Hence $z = Pz = Qz = Tz$, showing that z is a common fixed point of P , Q and T

Now, we prove the **uniqueness** of the common fixed point. If possible, let z' be another common fixed point of P , Q and T . Then from condition (ii), we have

$$(2. 2. 13) S(z, z', z') = S(Pz, Qz', Qz') \leq \Phi(\lambda(z, z')). \text{ But}$$

$$\lambda(z, z') = \max \{S(Tz, Tz', Tz'), S(Pz, Tz, Tz), S(Qz', Tz', Tz'),$$

$$\frac{1}{2} [S(Pz, Tz', Tz') + S(Qz', Tz, Tz)]\}$$

$$= S(z, z', z'). \text{ Therefore (2. 2. 13) gives}$$

(2. 2. 14) $S(z, z', z') \leq \Phi(S(z, z', z'))$ and this will be contradiction if $z \neq z'$. Therefore $z = z'$. Thus z is the unique common fixed point of P , Q and T .

Thus the Theorem 2. 1 is completely proved.

2.3 A Common Fixed Point Theorem for Three Selfmaps of a Complete S- metric space:

Before we prove the main result of this section, we prove the following lemma:

2.3.1 Lemma: Let (X, S) be a S- metric space and P , Q and T be selfmaps of X such that

$$(i) P(X) \cup Q(X) \subseteq T(X)$$

$$(ii) S(Px, Qy, Qy) \leq c. \lambda(x, y) \text{ for all } x, y \in X$$

where $0 \leq c < 1$ and $\lambda(x, y)$ is as defined in (ii)' of Theorem 2. 1

Further, if

$$(i) (X, S) \text{ is complete.}$$

Then for any $x_0 \in X$ and for any of its associated sequence $\{x_n\}$ relative to the three selfmaps, the sequence $Px_0, Qx_1, Px_2, Qx_3, \dots, Px_{2n}, Qx_{2n+1}, \dots$ converges to some $z \in X$.

Proof: Suppose P , Q and T be selfmaps of a S-metric space (X, S) for which the conditions (i) and (ii) hold. Let $x_0 \in X$ and $\{x_n\}$ be an associated sequence of x_0 relative to three selfmaps. Then, since $Px_{2n} = Tx_{2n+1}$ and $Qx_{2n+1} = Tx_{2n+2}$ for $n \geq 0$. Note that

$$\begin{aligned}\lambda(x_{2n}, x_{2n+1}) &= \max \{S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), S(Px_{2n}, Tx_{2n}, Tx_{2n}), S(Qx_{2n}, Tx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2} [S(Px_{2n}, Tx_{2n+1}, Tx_{2n+1}) + S(QTx_{2n+1}, Tx_{2n}, Tx_{2n})]\} \\ &= \max \{S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), S(Tx_{2n+1}, Tx_{2n}, Tx_{2n}), S(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}), \\ &\quad \frac{1}{2} [S(Tx_{2n+1}, Tx_{2n+1}, Tx_{2n+1}) + S(Tx_{2n+2}, Tx_{2n}, Tx_{2n})]\} = \max \{S(Tx_{2n}, Tx_{2n+1}, \\ &\quad Tx_{2n+1}), S(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}), \frac{1}{2} S(Tx_{2n+2}, Tx_{2n}, Tx_{2n})\}\end{aligned}$$

$$\begin{aligned}\text{Since } \frac{1}{2} S(Tx_{2n+2}, Tx_{2n}, Tx_{2n}) &= \frac{1}{2} [S(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}) + S(Tx_{2n+1}, Tx_{2n}, Tx_{2n})] \\ &\leq \max \{S(Tx_{2n+2}, Tx_{2n+1}, Tx_{2n+1}), S(Tx_{2n+1}, Tx_{2n}, Tx_{2n}),\end{aligned}$$

$$\lambda(x_{2n}, x_{2n+1}) \leq \max \{S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2})\}$$

Now by (ii)

$$\begin{aligned}S(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}) &= S(Px_{2n}, Qx_{2n+1}, qx_{2n+1}) \leq c \cdot \lambda(x_{2n}, x_{2n+1}) \\ &\leq c \cdot \max \{S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2})\}.\end{aligned}$$

Since $0 \leq c < 1$, it follows from that the

$$\max \{S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}), S(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2})\} = S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1})$$

$$\text{Therefore } S(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}) \leq c \cdot S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \dots\dots\dots (A)$$

Similarly, we can prove

$$S(Tx_{2n}, Tx_{2n+1}, Tx_{2n+1}) \leq c \cdot S(Tx_{2n}, Tx_{2n-1}, Tx_{2n-1}) \dots\dots\dots (B)$$

From (A) and (B), we get

$$\begin{aligned}S(Tx_{2n+1}, Tx_{2n+2}, Tx_{2n+2}) &\leq c^2 S(Tx_{2n}, Tx_{2n-1}, Tx_{2n-1}) \\ &\leq c^4 S(Tx_{2n-1}, Tx_{2n-3}, Tx_{2n-3}) \\ &\quad \begin{array}{c} - \text{-----} \\ - \text{-----} \end{array} \\ &\leq c^{2n} S(Tx_2, Tx_0, Tx_0) \rightarrow 0\end{aligned}$$

Since $c^{2n} \rightarrow 0$ as $n \rightarrow \infty$ (because $c < 1$), the sequence $\{Tx_n\}$ and hence $Px_0, Qx_1, Px_2, Qx_3, \dots, Px_{2n}, Qx_{2n+1}, \dots$ is a Cauchy sequence in the complete space (X, S) and therefore converges to a point say $z \in X$, proving lemma.

2.3.2 Remark: The converse of lemma is not true. That is, suppose P, Q and T are selfmaps of a S -metric space (X, S) satisfying condition (i) and (ii) of Lemma 2.3.1. Even, if for each $x_0 \in X$ and for each associated sequence $\{x_n\}$ of x_0 relative to P, Q and T , the sequence $Px_0, Qx_1, Px_2, Qx_3, \dots, Px_{2n}, Qx_{2n+1}, \dots$ converges in X , then (X, S) need not complete.

2.3.3 Theorem: Suppose (X, S) is a S -metric space satisfying conditions (i) to (iv) of Theorem 2.1. Further, if $(v)'$ (X, S) is complete

then P, Q and T have a unique common fixed point $z \in X$.

Proof: In view of Lemma 2.3.1 the condition (v) of Theorem 2.1 holds as view of (v)'.

Hence the Theorem follows from Theorem 2.1.

2.3.4 Corollary ([9]): Let P, Q and T be selfmaps of a metric space (X, d) such that

$$(i) \quad P(X) \cup Q(X) \subseteq T(X)$$

$$(ii) \quad d(Px, Qy) \leq c \lambda_0(x, y) \text{ for all } x, y \in X,$$

where

$$(ii)' \quad \lambda_0(x, y) = \max \{d(Tx, Ty), d(Px, Tx), d(Qy, Ty), \frac{1}{2}[d(Px, Ty) + d(Qy, Tx)]\} \text{ and } 0 \leq c < 1$$

$$(iii) \quad T \text{ is continuous,}$$

and

$$(iv) \quad PT = TP \text{ and } QT = TQ$$

Further, if

$$(v) \quad X \text{ is complete}$$

Then P, Q and T have a unique common fixed point in $z \in X$.

Proof: Given (X, d) is a metric space satisfying condition (i) to (v) of the corollary. If $S_1(x, y, z) = \max \{d(x, y), d(y, z), d(z, x)\}$, then (X, S_1) is a S-metric space and $S_1(x, y, x) = d(x, y)$. Therefore condition (ii) can be written as $S_1(Px, Qy, Qy) \leq c \lambda(x, y)$ for all $x, y \in X$ where $\lambda(x, y) = \max \{S_1(Tx, Ty, Ty), S_1(Px, Tx, Tx), S_1(Qy, Jy, Jy),$

$$\frac{1}{2}[S_1(Px, Ty, Ty) + S_1(Qy, Tx, Tx)]\}$$

which is the same as condition (ii) of Theorem 2.3.3. Also since (X, d) is complete, we have (X, S_1) is complete by Corollary 1.14.

Now, P, Q and T are selfmaps on (X, S_1) satisfying conditions of Theorem 2.3.3 and hence the corollary follows.

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